1. NUMERAL SYSTEMS

There are three basic concepts of a number representation:

- sign-value notation,
- subtractive sign-value notation,
- positional notation (place-value notation).

A ***sign-value notation*** represents numbers by a series of numerals whose sum show up the represented number. In the Ancient Egyptian numeral system, for example, 𓍢 means hundred and 𓎆 means ten (Tab. 1.1), so 𓍢𓍢𓍢𓎆 means three hundred and ten (100 + 100 + 100 + 10).

Tab. 1.1. Ancient Egyptian numerals

Value	1	10	100	1000
Hieroglyph	𓏺	𓎆	𓍢	𓆼
Description	Single stroke	Heel bone	Coil of rope	Water lily (Lotus)

Value	10 000	100 000	1 000 000
Hieroglyph	𓂭	𓆐	𓁨
Description	Finger	Tadpole or Frog	Man with both hands raised

Egyptian hieroglyphs were written in both directions (and even vertically). The main disadvantage of the Ancient Egyptian numeral system was that large numbers require a lot of hieroglyphs. For example, a large number like 999 needs 27 hieroglyphs (𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺).

Acrophonic Greek numerals

A little bit shorter form of sign-value notation was based on Acrophonic Greek numerals[1], innovated in the first millennium BC. Beside the symbols for 1, 10, 100, 1000 and 10000, the system had intermediate symbol for 5 and compound symbols for 50, 500, 5000 and 50000. The compound symbols were made by combining the symbol 5 with the symbols 10, 100, 1000 and 10000, Tab. 1.2.

Tab. 1.2. Acrophonic Greek numerals

Value	1	5	10	5·10	100	5·100	1000
Symbol	Ι	Γ or Π	Δ	𐅄	Η	𐅅	Χ
Value		5·1000	10 000	5·10 000	Example: 2064		
Symbol		𐅆	Μ	𐅇	ΧΧ𐅄ΔΙΙΙΙ		

Alphabetic Greek numerals

Further improvements in a sign-value notation were achieved in the fifth century BC in the Alphabetic Greek numerals. Instead of using the separate set of symbols for numbers, values were assigned to lowercase letters of the old Greek alphabet based on their native alphabetic order, Tab. 1.3.

Tab. 1.3. Values assigned to symbols in the old Greek alphabet

Value	1	2	3	4	5	6	7	8	9
Symbol	α	β	γ	δ	ε	ς	ζ	η	ϑ
Value	10	20	30	40	50	60	70	80	90
Symbol	ι	κ	λ	μ	ν	ξ	ο	π	ϙ
Value	100	200	300	400	500	600	700	800	900
Symbol	ρ	ϲ	τ	υ	φ	χ	ψ	ω	ϡ
Value	1000	2000	3000	4000	5000	Example: 2064			
Symbol	,α	,β	,γ	,δ	,ε	,βξδ’			

[1] The reason why this system is called *acrophonic* is because the numerals for 5, 10, 100, 1000 are the first letters of the Greek words for these numbers, namely *ΠΕΝΤΕ*, *ΔΕΚΑ*, *HEKATON*, and *ΧΙΛΙΟΝ*.

Of the 27 letters, nine were for units (1, 2, …, 9), nine for tens (10, 20, …, 90) and nine for hundreds (100, 200, …, 900). To mark that a sequence of letters is in fact a number (and not a text), a special sign like a vertical dash or accent assign is placed after the numerals, i.e., σνζ′ which represents 200+50+7=257. To write numbers larger than 1000, a similarly vertical dash sign is placed before the numerals and below the line of writing, such as ͵αϡπϑ′ representing 1000+900+80+9=1989.

Glagolitic numerals

The Glagolitic[1] numeral system is similar to the Alphabetic Greek numeral system. To Glagolitic letters were assigned values based on their native alphabetic order, Tab. 1.4. Nine letters were for units (1, 2, …, 9), nine for tens (10, 20, …, 90), nine for hundreds (100, 200, …, 900) and remainder for thousands (1000, 2000,…). Numbers were distinguished from text by small square marks ♦, one before and one after each symbol.

Tab. 1.4. Values assigned to symbols in the Glagolitic alphabet[2]

Value	1	2	3	4	5	6	7	8	9
Symbol	Ⰰ	Ⰱ	Ⰲ	Ⰳ	Ⰴ	Ⰵ	Ⰶ	Ⰷ	Ⰸ
Value	10	20	30	40	50	60	70	80	90
Symbol	Ⰹ	Ⰻ	Ⰼ	Ⰽ	Ⰾ	Ⰿ	Ⱀ	Ⱁ	Ⱂ
Value	100	200	300	400	500	600	700	800	900
Symbol	Ⱃ	Ⱄ	Ⱅ	Ⱆ	Ⱇ	Ⱈ	Ⱉ	Ⱋ	Ⱌ
Value	1000	2000	3000	4000	5000	Example: 2064			
Symbol	Ⱍ	Ⱎ	Ⱏ	Ⱑ	Ⱓ	♦Ⱎ♦Ⰿ♦Ⰳ♦			

[1] The Glagolitic is the oldest known Slavic alphabet. It was invented during the 9th century by the Byzantine missionaries St Cyril (827-869 AD) and St Methodius (826-885 AD) in order to translate the Bible and other religious works into the language of the Great Moravia region. It is probably modelled on a cursive form of the Greek alphabet while their translations are based on a Slavic dialect of the Thessalonika area, which formed the basis of the literary standard known as Old Church Slavonic. This old Slavic scripts had remained in use by Croats up to 19th century.

[2] The Croation version of symbols which was used about 14th century .

For example, Ⰰ denotes a letter, while ⬥Ⰰ⬥ denotes the numeral “1”. If needed, to represent numbers which are not assigned to any letter, two or more symbols would have to be adjoined together. For example, the number 2014 may be written as 2000+4+10 = ⬥Ⱎ⬥Ⰳ⬥Ⰹ⬥.

Subtractive sign-value notation
Some improvements were made by *subtractive sign-value notation.* Roman numerals, for example, are generally written in descending order from left to right, but where a symbol of a smaller value precedes a symbol of a larger value, a smaller value is subtracted from a larger value, and the result is added to the total[1]. For example, M means thousand and I means one (Tab. 1.5), so MI denotes number $1000+1=1001$, while IM denotes number $-1+1000=999$.

Tab. 1.5. Roman numerals[2]

Value	1	5	10	50	100	500	1000
Symbol	I	V	X	L	C	D	M
Value		5 000	10 000	50 000	100 000	500 000	1 000 000
Symbol		$\overline{\text{V}}$	$\overline{\text{X}}$	$\overline{\text{L}}$	$\overline{\text{C}}$	$\overline{\text{D}}$	$\overline{\text{M}}$

Note that there is no need for the zero in a sign-value notation.

Sign-value notation was the pre-historic way of writing numbers and only gradually evolved into the *positional notation*, also known as *place-value notation*, in which the value of a particular digit depends both on the digit itself and its position within the number.

Positional notation
In the *positional notation*, a number is represented by a sign (plus or minus) and digits, while the digital coma or the digital point (depending on a country) separates an integer part of a number from its fraction. Unlike the sign-value

[1] The notation of Roman numerals has varied through the centuries. Originally, it was common to use IIII instead of IV to represent four, because IV are the first two letters and consequently abbreviation for IVPITER, the Latin script spelling for the Roman god Jupiter. The notation which uses IV instead of IIII has become the standard notation in modern time, but with some exceptions. For example, Louis XIV, the king of France, who preferred IIII over IV, ordered his clockmakers to produce clocks with IIII and not IV, and thus it has remained [1].

[2] Roman numerals have remained in use mostly for the notation of *Anno Domini* years, and for numbers on clockfaces. Sometimes, Roman numerals are still used for enumerating the lists (as an alternative to alphabetical enumeration), and for numbering pages in prefatory material in books.

notation, there is an explicit need for zero in the positional notation. The number of digits with integer values zero, one, two, … that form a positional numeral system is called the ***base of the numeral system*** (e.g. if there are 10 digits with values from zero to nine, the base is 10).

An integer number can be represented in a positional numeral system in base B with a sum of digits $d_k \in \{\text{zero}, \text{one}, ..., B-1\}$ multiplied by powers k of the base B as

$$inumber = (d_n ... d_1 d_0)_B = d_n \cdot B^n + ... + d_1 \cdot B^1 + d_0 \cdot B^0 . \qquad (1.1)$$

A real number can be represented in base B as

$$rnumber = (d_n ... d_0, d_{-1} ...)_B = d_n \cdot B^n + ... + d_0 \cdot B^0 + d_{-1} \cdot B^{-1} + ... \qquad (1.2)$$

In both cases the leading zeroes are usually omitted assuming that d_n is first non-zero digit.

Sexsagesimal system
Possibly the oldest positional numeral system is a ***sexsagesimal system***, used around 3100 B.C., in Babylon. It is a combination of the sign-value and the positional notation in base 60 [2].

In Babylon, digits up to 59 were noted by using two symbols in the sign-value notation: symbol 𒁹 to count units and symbol 𒌋 to count tens. These symbols 𒁹 and 𒌋 and their values were combined to form 59 digits in a notation similar to that of Roman numerals; for example, the combination 𒌋𒌋𒐈 represented the digit with a value of 23 (Fig. 1.1).

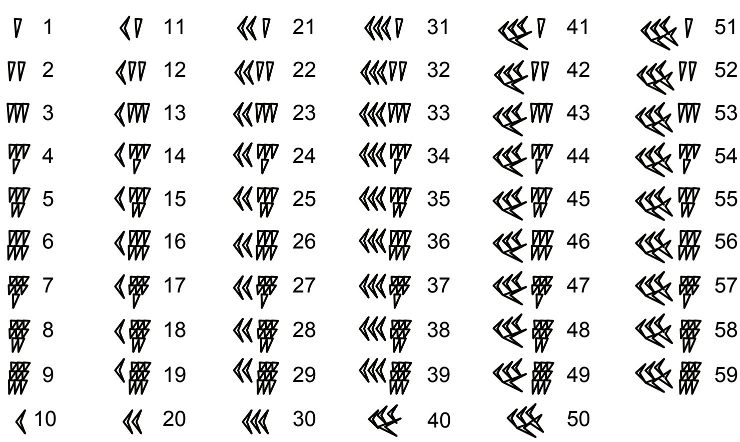


Fig. 1.1. *Digits used in Babylonian numeral system*

These 59 digits are then used in positional numeral system in base 60 as it is illustrated in Example 1.1.

Example 1.1. Babylonian numerals:

$$
\begin{aligned}
&𒎙𒐈 &&= 10+10+1+1+1 = 23\,, \\
&𒌍 &&= 10+10+10 = 30\,, \\
&𒈫 &&= 1+1 = 2\,, \\
&𒎙𒈫\ 𒁹 &&= ("22"\,"1")_{60} = 22\cdot 60^1 + 1\cdot 60^0\,, \\
&𒈫\ 𒌍 &&= ("2"\,"30")_{60} = 2\cdot 60^1 + 30\cdot 60^0.
\end{aligned} \tag{1.3}
$$

The Babylonians did not have a digit for zero. What the Babylonians used instead was a space (and later a disambiguating placeholder symbol 𒑱) to indicate a place without value, similar to zero.

In addition, Babylonians did not have any mark to separate integer from the fractional part of a number, but they calculated with real numbers, as it is illustrated in Example 1.2.

Example 1.2. The Babylonian approximation of the square root of 2 in the context of Pythagoras' theorem for an isosceles triangle.

The approximation is illustrated on the round tablet that was, we believe, an education tablet from Ancient Babylonia, dated 1800 B.C. [3]. The tablet, Fig. 1.2, has a square with both diagonals drawn in. On one side of the square is written 𒌍 $= 30$, the length of the square side. If this is treated as a fraction of the number (i.e., as the first figure of the fraction), then

$$𒌍 = 30\cdot 60^{-1} = 1/2\,. \tag{1.4}$$

Along one of the diagonals, the number is written

$$𒁹\ 𒎙𒐉\ 𒐐𒁹\ 𒌋 \;=\; 1 + 24/60 + 51/60^2 + 10/60^3 = 1{,}41421296... \approx \sqrt{2} \tag{1.5}$$

and below it is the number

$$𒐏𒈫\ 𒎙𒐊\ 𒌍𒐊 = 42/60 + 25/60^2 + 35/60^3 = 0{,}70710648... \approx \sqrt{1/2}\,. \tag{1.6}$$

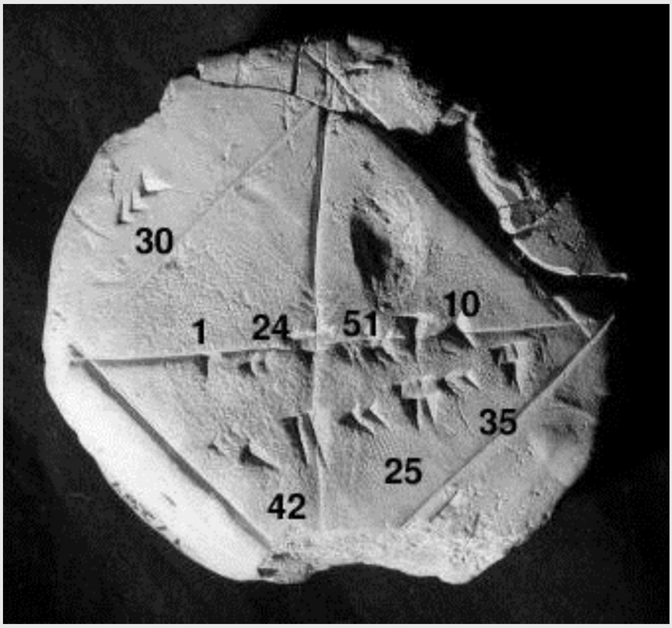


Fig. 1.2. *Round tablet illustrating approximation of square root* [3]

In fact, the number 𒁹 𒎙𒐉 𒐐𒁹 𒌋 is a remarkably good approximation of $\sqrt{2} = 1{,}41421356\ldots$ in four sexagesimal figures, which is about six decimal figures. Moreover, it is easy to see that 𒁹 𒎙𒐉 𒐐𒁹 𒌋 $\approx \sqrt{2}$ multiplied by $1/2$ is 𒐏𒐖 𒎙𒐊 𒌍𒐊 $\approx \sqrt{1/2}$, the diagonal length of the square of the side 𒌍$= 1/2$.

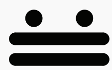
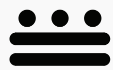
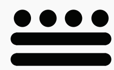
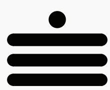
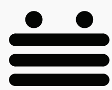
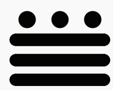
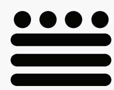
Hexagesimal system

This system must be distinguished from the sexsagesimal system, although both systems have the same base (and that is 60). One digit in the hexagesimal system is a decimal number from 0 to 59. Today, the hexagesimal system is used to express time (hours, minutes and seconds).

Vigesimal system

Another interesting positional system is a *vigesimal system* (base-twenty) used by the Pre-Columbian Maya civilization. The Maya numerals were made up by combining three symbols in signed value notation: zero (shell shape 𝋠), one (a dot •) and five (a bar —). The construction of 20 numerals (starting from zero) is shown in Tab. 1.6. For example, seventeen (𝋱) is written as two dots in a horizontal row above three horizontal bars stacked upon each other.

Tab. 1.6. Digits used in numeral system of Pre-Columbian Maya

Value	0	1	2	3	4
Glyph					
Value	5	6	7	8	9
Glyph					
Value	10	11	12	13	14
Glyph					
Value	15	16	17	18	19
Glyph					

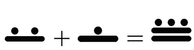
Because the base of the numeral system was 20, larger numbers were written down in powers of 20 from bottom to top. Fig. 1.3 shows how the number 2402 was written.

$$6 \cdot 20^2 = 2400$$
$$0 \cdot 20^1 = \quad 0$$
$$2 \cdot 20^0 = \quad 2$$
$$\text{sum} = 2402$$

Fig. 1.3. *Number 2402 in Maya numeral system*

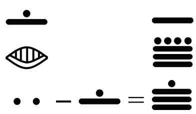
As it can be seen, the addition is just a matter of adding up dots and bars. Maya merchants often used cocoa beans, which they laid out on the ground to do these calculations.

Addition is performed by combining the numeric symbols at each level

 $\quad (7 + 6 = 13)$. (1.7)

If five or more dots result from the combination, then five dots are replaced by a bar. If four or more bars result from the combination, then four bars are replaced by a shell and a dot is added to the next higher row.

In subtraction, elements of the subtrahend symbol are removed from the minuend symbol:



$$(2402 - 6 = 2396 = 5 \cdot 20^2 + 19 \cdot 20^1 + 16 \cdot 20^0). \qquad (1.8)$$

If there are not enough dots in a minuend position, then a bar is replaced by five dots. If there are not enough bars, then a dot is removed from the next higher minuend symbol in the column and four bars are added to the minuend symbol being worked on. Note that this corresponds almost exactly to the traditional addition and subtraction in the common base 10.

Counting rods

As one of ancient decimal systems, *counting rods* were used by mathematicians for calculation in ancient China, Japan, Korea, and Vietnam for more than two thousand years. They are small bars, typically 3-14 cm long and placed either horizontally or vertically (Tab. 1.7). The written forms based on them are called ***rod numerals***. That was a true positional numeral system with digits for 1 to 9, and later for 0.

Tab. 1.7. Counting roads (traditional version)

Vertical	Value	0	1	2	3	4
	Symbol	〇	𝍠	𝍡	𝍢	𝍣
	Value	5	6	7	8	9
	Symbol	𝍤	𝍥	𝍦	𝍧	𝍨
Horizontal	Value	0	1	2	3	4
	Symbol	〇	𝍩	𝍪	𝍫	𝍬
	Value	5	6	7	8	9
	Symbol	𝍭	𝍮	𝍯	𝍰	𝍱

Positive numbers are written as they really are and the negative numbers are written with a slant bar at the last digit. The vertical bar in the horizontal forms 6-9 is drawn shorter to have the same character height.

Addition is performed by combining the numeric symbols

𝍦 + 𝍥 = 𝍠𝍢 or 𝍯 + 𝍮 = 𝍩𝍫 $(7+6=13)$. (1.9)

Subtraction is performed by adding a negative number

𝍦 + 𝍥̸ = 𝍠 or 𝍯 + 𝍮̸ = 𝍩 $(7+-6=1)$. (1.10)

Modern numeral systems

In the modern mathematics, when used together with the Latin alphabet, digits are read from left to right. They are numerals 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, usually referred to as Arabic numerals. Digits, whose value is greater than 9, are usually represented by lowercase letters a, b, c, … or capital letters A, B, C, … respectively (Chapter 1.4). The digits are always written uprightly (not as *italic*).

As befitting their history, the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are more appropriately known as Hindu or Hindu-Arabic numerals. The reason that they are more commonly known as "Arabic numerals" in Europe and the Americas is that they were introduced to Europe in the 10th century from Arabs of North Africa, who were then using the digits from Libya to Morocco. Europeans did not know about the roots of numerals in ancient India, so they named them "Arabic numerals". Arabs, on the other hand, call the system "Hindu numerals", referring to their origin in India[1].

The decimal Hindu-Arabic numeral system was invented in India[2] around 500 AD [4]. The system was revolutionary in that it included a zero and positional notation. Fibonacci, a mathematician born in the Republic of Pisa, Italia, who had studied in Bejaia (Bougie), Algeria, promoted the Indian numeral system in Europe with his book *Liber Abaci*, which was written in 1202. The European

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[1] This is not to be confused with what the Arabs call the "Hindi numerals", namely the Eastern Arabic numerals (٠,١,٢,٣,٤,٥,٦,٧,٨,٩) used in the Middle East, or any of the numerals currently used in Indian languages (e.g. Devanagari: ०.१.२.३.४.५.६.७.८.९) [4].

[2] In 825 Al-Khwārizmī wrote a treatise in Arabic, *On the Calculation with Hindu Numerals*, which was translated to Latin from Arabic in the 12th century as *Algoritmi de Numero Indorum*, where Algoritmi, the translator's rendition of the author's name, gave rise to the word *algorithm* (Latin *algorithmus*, "calculation method").