1. INTRODUCTION TO INTERPOLATION

Interpolation is the process of defining an *interpolation function* $\mathcal{F}(x)$ that (and possibly its derivatives $\mathcal{F}'(x)$, $\mathcal{F}''(x)$, …) takes on specified values at a specified set of **distinct** sampling points. These sampling points are usually called *interpolation nodes*, or just *nodes*. The space between end-nodes is called the *domain of interpolation*, or just *domain*, Fig. 1.1.

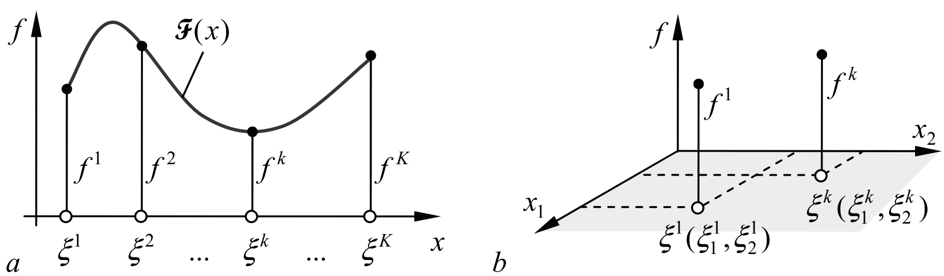

Fig. 1.1. *Sampling points, a – in the one-dimensional domain, b – in the two-dimensional domain*

In some interpolation methods, the distribution of nodes is arbitrary, in some it is predetermined by the method. **It should be emphasised that the number of nodes as well as their distribution significantly influences Runge's phenomenon and interpolation error** (Chapter 2.4).

The interpolation function $\mathcal{F}(x)$ is a function of coordinates $x_1, x_2,...$ of a point x at the domain of interpolation. If the domain is one-dimensional, then *x* denotes both a point and its coordinate. The same notation is used for nodes ξ^k, Fig 1.1.

Regarding their level, interpolation may be referred to as

- simple interpolation (by using sampled values of $\mathcal{F}(x)$) and
- higher level interpolation (by using sampled values of $\mathcal{F}(x)$, $\mathcal{F}'(x)$, …).

Both types of interpolation are described in the following sections.

1.1 SIMPLE INTERPOLATION

The interpolation function $\mathcal{F}(x)$ takes on *K* values f^k at *K* distinct sampling points ξ^k ($k = 1,2,...,K$), Fig. 1.1*a*. This condition can be written as

$$(\forall k = 1,...,K) \quad \mathcal{F}(\xi^k) = f^k \,. \tag{1.1}$$

Interpolation function $\mathcal{F}(x)$ is usually defined by one of the two following interpolation formulas

$$\mathcal{F}(x) = \sum_{n=0}^{K-1} \beta_n \phi_n(x) \,, \tag{1.2}$$

$$\mathcal{F}(x) = \sum_{k=1}^{K} f^k N^k(x) \,. \tag{1.3}$$

In the first interpolation formula (1.2), interpolation function $\mathcal{F}(x)$ is a linear combination of the so called *coordinate functions* $\phi_n(x)$ ($n = 0, 1, ..., K-1$). The coordinate functions may be chosen arbitrarily as long as their set is ***complete*** and ***linearly independent***, while interpolation coefficients β_n are determined by *K* values f^k of interpolation functions $\mathcal{F}(x)$ at *K* distinct sampling points ξ^k ($k = 1,2,...,K$) and by the distribution of these points.

In the second interpolation formula (1.3), the interpolation function $\mathcal{F}(x)$ is a linear combination of the so called *shape functions* $N^k(x)$ ($k = 1, ..., K$). The name *shape function* has origin in the finite elements method, where it is used not only for interpolation but also in determining the shape of finite elements.

Shape functions are determined only by a distribution of *K* distinct sampling points ξ^k ($k = 1,2,...,K$). It can be also interpreted as a unique set of coordinate functions whose interpolation coefficients β_n take values $f^k = \mathcal{F}(\xi^k)$.

Simple interpolation provides the same results as an approximation of function on a discrete domain in which all available elements of the finite approximation series are employed in calculation[1].

1.1.1 Coordinate functions

A set of coordinate functions $\phi_n(x)$ must be ***complete*** and ***linearly independent.***

Complete set of coordinate functions

A chosen set of coordinate functions $\phi_n(x)$ is *complete* if the interpolation function $\mathcal{F}(x)$ can satisfy any set of specified interpolation data.

Complete sets of coordinate functions, for example, are

$$1, x, x^2, x^3, \ldots \quad \text{and} \quad 1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \ldots . \tag{1.4}$$

The sets of coordinate functions

$$x, x^2, x^3, \ldots \quad \text{and} \quad \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \ldots . \tag{1.5}$$

are incomplete because they cannot satisfy the condition $\mathcal{F}(\xi) \neq 0$ at node $\xi = 0$.

However, in some cases a complete set of coordinate functions is reduced to a subset that is complete with respect to additional conditions. For example, the subsets of coordinate functions

$$1, x^2, x^4, \ldots \quad \text{and} \quad 1, \cos x, \cos 2x, \ldots \tag{1.6}$$

are complete if the interpolation function $\mathcal{F}(x) = \mathcal{F}(-x)$ is to be even, while the subsets of coordinate functions

$$x, x^3, x^5, \ldots \quad \text{and} \quad \sin x, \sin 2x, \sin 3x, \ldots \tag{1.7}$$

are complete if the interpolation function $\mathcal{F}(x) = -\mathcal{F}(-x)$ is to be odd.

Linearly independent coordinate functions

The number of linearly independent coordinate functions $\phi_n(x)$ (and coefficients β_n) in the interpolation function $\mathcal{F}(x)$ must match the total number of samples.

If the chosen set of coordinate functions $\phi_n(x)$ is linearly dependent, then at least one coordinate function $\phi_L(x)$ is redundant and the term $\beta_L \phi_L(x)$ should be removed from the interpolation function $\mathcal{F}(x)$.

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[1] Approximations of functions are described in Volume III of *Numerical Methods*.

If, for example, there is a linearly dependent coordinate function

$$\phi_L(x) = \sum_{\substack{n=0 \\ n \neq L}}^{N-1} \alpha_n \phi_n(x) , \tag{1.8}$$

the interpolation function $\mathcal{F}(x)$ may be transformed into

$$\mathcal{F}(x) = \sum_{\substack{n=0 \\ n \neq N}}^{N-1} \beta_n \phi_n(x) + \beta_L \sum_{\substack{n=0 \\ n \neq L}}^{N-1} \alpha_n \phi_n(x) , \tag{1.9}$$

and finally reduced into

$$\mathcal{F}(x) = \sum_{\substack{n=0 \\ n \neq N}}^{N-1} (\beta_n + \beta_L \alpha_n)\, \phi_n(x) \quad \text{or} \quad \mathcal{F}(x) = \sum_{\substack{n=0 \\ n \neq N}}^{N-1} \overline{\beta}_n \phi_n(x) . \tag{1.10}$$

The number of interpolation coefficients $\overline{\beta}_n$ is reduced. Therefore, any linearly dependent function $\phi_L(x)$ is redundant and has to be omitted.

Coordinate functions in polynomial interpolation
In most cases coordinate functions are powers of coordinates

$$\phi_k = x^k \quad (k = 0,...,K-1), \tag{1.11}$$

or even a set of polynomials

$$\phi_k = \sum_r a_{kr} x^r \quad (k = 0,...,K-1). \tag{1.12}$$

The interpolation function $\mathcal{F}(x)$ is also a polynomial. Interpolation with powers of coordinates or with polynomials of any form is called *polynomial interpolation*. Various methods of polynomial interpolation are described in Chapter 2.

Coordinate functions in non-polynomial interpolation
Whereas polynomial interpolation is usually most suitable when applied on a finite interpolation interval, non-polynomial interpolation must be used when interpolation interval is infinite or semi-infinite. Coordinate functions can be exponential, logarithmic, rational, trigonometric, etc. Various methods of non--polynomial interpolation are described in Chapter 3.

Coordinate functions on multidimensional domain

Let a function $\mathcal{F}(x) = f(x_1, x_2, ..., x_M)$ of M coordinates x_1, x_2, …, x_M be determined in the M-dimensional domain by its values $f^k = f(\xi^k)$ at K distinct sampling points $\xi^k(\xi_1^k, \xi_2^k, ..., \xi_M^k)$. Iinterpolation can be performed by the interpolation formula

$$\mathcal{F}(x_1, x_2, ..., x_M) = \sum_{n=0}^{K-1} \beta_n \phi_n(x_1, x_2, ..., x_M). \tag{1.13}$$

Herein, $\{\phi_n(x)\} = \{\phi_n(x_1, x_2, ..., x_M)\}$ is assumed to be a complete set of linearly independent *coordinate functions* of M coordinates x_1, x_2, …, x_M. The K unknown interpolation coefficients β_n are determined by the condition $\mathcal{F}(\xi^k) = f^k$ (1.1).

However, this is a general case in which nodes ξ^k may be distributed randomly on the M-dimensional interpolation domain.

Sampling points may be arranged in the form of a mesh, as those shown in Fig. 1.2.

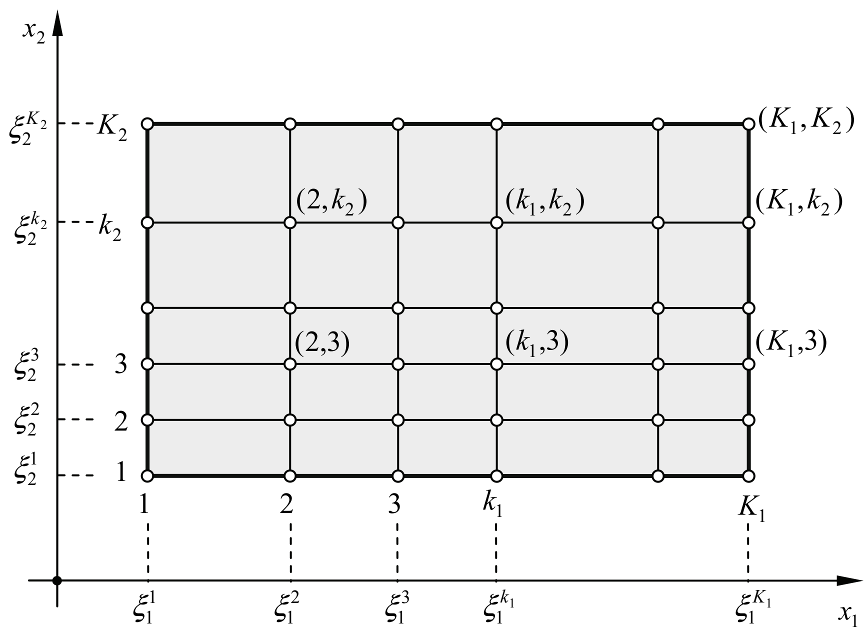

Fig. 1.2. *Example of a mesh of sampling points in a 2D domain*

In such a case, the coordinate functions $\phi_n(x_1, x_2, ..., x_M)$ of *M*-variables can be factorised into the product of coordinate functions $\phi_{n_r}^{[r]}(x_r)$ of a single variable

$$\phi_{n(n_1,n_2,...,n_M)}(x_1, x_2, ..., x_M) = \phi_{n_1}^{[1]}(x_1)\phi_{n_2}^{[2]}(x_2)\cdots\phi_{n_M}^{[M]}(x_M)\ . \tag{1.14}$$

Interpolation can be provided by the formula

$$\begin{aligned}&\mathcal{F}(x_1, x_2, ..., x_M) = \\ &= \sum_{n_1=0}^{K_1-1}\sum_{n_2=0}^{K_2-1}\cdots\sum_{n_M=0}^{K_M-1}\beta_{n_1 n_2 ... n_M}\phi_{n_1}^{[1]}(x_1)\phi_{n_2}^{[2]}(x_2)\cdots\phi_{n_M}^{[M]}(x_M)\,.\end{aligned} \tag{1.15}$$

This simplification can be done for some classes of coordinate functions; e.g., in a polynomial interpolation.

1.1.2 Interpolation coefficients in the simple interpolation

Interpolation coefficients β_n are determined by *K* values f^k of interpolation functions $\mathcal{F}(x)$ at *K* distinct sampling points ξ^k ($k = 1,2,...,K$) and by distribution of these points itself.

The interpolation formula (1.2) then results an a linear equation system

$$(\forall k = 1, 2, ..., K)\quad \mathcal{F}(\xi^k) \equiv \sum_{n=0}^{K-1}\beta_n\phi_n(\xi^k) = f^k\ . \tag{1.16}$$

This equation system can be expressed with matrices as

$$[\phi_n(\xi^k)]\{\beta_n\} = \{f^k\} \tag{1.17}$$

or

$$\begin{bmatrix}\phi_0(\xi^1) & \phi_1(\xi^1) & ... & \phi_{K-1}(\xi^1)\\ \phi_0(\xi^2) & \phi_1(\xi^2) & ... & \phi_{K-1}(\xi^2)\\ ... & ... & & ...\\ \phi_0(\xi^K) & \phi_1(\xi^K) & ... & \phi_{K-1}(\xi^K)\end{bmatrix}\begin{Bmatrix}\beta_0\\ \beta_1\\ ...\\ \beta_{K-1}\end{Bmatrix} = \begin{Bmatrix}f^1\\ f^2\\ ...\\ f^K\end{Bmatrix}. \tag{1.18}$$

The solution

$$\{\beta_n\} = [\phi_n(\xi^k)]^{-1}\{f^k\}\,, \tag{1.19}$$

is a set of unknown interpolation coefficients β_n. A simple polynomial interpolation at three sampling points is illustrated in Example 1.1.

Example 1.1. A simple polynomial interpolation with three sampling points.

Suppose that interpolation function $\mathcal{F}(x)$ takes on values

$$\mathcal{F}(0) = 12\,,\ \mathcal{F}(1) = 6\ \text{ and }\ \mathcal{F}(3) = 0\,. \tag{1.20}$$

By using the coordinate functions $\phi_0 = 1$, $\phi_1 = x$ and $\phi_2 = x^2$, polynomial interpolation may be performed by using polynomial

$$\mathcal{F}(x) = \beta_0 + \beta_1 x + \beta_2 x^2\,. \tag{1.21}$$

Interpolation coefficients β_n are determined by the equation system

$$\begin{aligned} \mathcal{F}(0) &\equiv \beta_0 = 12,\\ \mathcal{F}(1) &\equiv \beta_0 + \beta_1 + \beta_2 = 6,\\ \mathcal{F}(3) &\equiv \beta_0 + 3\beta_1 + 9\beta_2 = 0, \end{aligned} \tag{1.22}$$

whose solution is $\beta_0 = 12$, $\beta_1 = -7$, $\beta_2 = 1$. Interpolation function is the polynomial

$$\mathcal{F}(x) = 12 - 7x + x^2\,. \tag{1.23}$$

By appropriate choice of coordinate functions $\phi_k(x)$, the matrix of linear equation system (1.18) becomes triangular (Newton interpolation, Chapter 2.1). Moreover, solving the linear equation system can be avoided completely, like in *Lagrange interpolation* (Chapter 2.2), *Chebyshev interpolations* (Chapter 2.3 and 3.4), *Hermite interpolation* (Chapter 2.5) or *Fourier interpolations* (Chapter 3.4).

1.1.3 Finding interpolation coefficients by Cramer's rule

Equation system (1.18) can be solved by applying Cramer's rule. Interpolation coefficients β_n are then the quotient of two determinants

$$\beta_n = D_n / D\,. \tag{1.24}$$

Determinant D is the determinant of the system matrix $[\phi_n(\xi^k)]$:

$$D = \left|\phi_n(\xi^k)\right| = \begin{vmatrix} \phi_0(\xi^1) & \phi_1(\xi^1) & \dots & \phi_{K-1}(\xi^1) \\ \phi_0(\xi^2) & \phi_1(\xi^2) & \dots & \phi_{K-1}(\xi^2) \\ \dots & \dots & & \dots \\ \phi_0(\xi^K) & \phi_1(\xi^K) & \dots & \phi_{K-1}(\xi^K) \end{vmatrix}\,. \tag{1.25}$$

Determinants D_n are obtained from determinant D by replacing the coefficients $\phi_n(\xi^1), \phi_n(\xi^2), ..., \phi_n(\xi^K)$ in the $n+1^{\text{st}}$ column with values $f^1, f^2, ..., f^K$ of the interpolation function $\mathcal{F}(x)$.

For example,

$$D_1 = \begin{vmatrix} f^1 & \phi_1(\xi^1) & ... & \phi_{K-1}(\xi^1) \\ f^2 & \phi_1(\xi^2) & ... & \phi_{K-1}(\xi^2) \\ ... & ... & & ... \\ f^K & \phi_1(\xi^K) & ... & \phi_{K-1}(\xi^K) \end{vmatrix}, \text{ etc.} \tag{1.26}$$

Determinant D_n can be expanded into

$$D_n = \sum_{k=1}^{K} (-1)^{k+n} G_{kn} f^k , \tag{1.27}$$

where G_{kn} are sub-determinants of the system matrix $[\phi_n(\xi^k)]$, obtained by removing coefficients in k^{th} row and coefficients in the $n+1^{\text{st}}$ column. Hence,

$$\beta_n = \frac{1}{D} \sum_{k=1}^{K} (-1)^{k+n} G_{kn} f^k . \tag{1.28}$$

1.1.4 Simple interpolation with orthogonal coordinate functions

Consider that functions $\phi_n(x)$ are orthogonal[1] to the given set of sampling points with respect to the weighting coefficients w^k. By multiplying both sides of each equation in a linear equation system (1.16) with $w^k \phi_m(\xi^k)$, and by providing summation over $k = 1, 2, ..., K$, it can be obtained that

$$(\forall m = 0, 1, ..., K-1)$$
$$\sum_{k=1}^{K} w^k f^k \phi_m(\xi^k) = \sum_{n=0}^{K-1} \left(\beta_n \sum_{k=1}^{K} w^k \phi_n(\xi^k) \phi_m(\xi^k) \right). \tag{1.29}$$

[1] Orthogonal functions and inner products are described in volume III of *Numerical Methods*.

This linear equation system can also be written in terms of *inner products* on a discrete domain as

$$(\forall m = 0,1,...,K-1) \;\; \langle f, \phi_m \rangle = \sum_{n=0}^{K-1} \beta_n \langle \phi_n, \phi_m \rangle \; . \tag{1.30}$$

Due to the presumption that functions $\phi_m(x)$ are orthogonal on the finite set of K distinct points with respect to the weighting coefficients w^k, all inner products $\langle \phi_n, \phi_m \rangle$ for $n \neq m$ vanish, finally giving

$$(\forall m = 0,1,...,K-1) \;\; \langle f, \phi_m \rangle = \beta_m \langle \phi_m, \phi_m \rangle \; . \tag{1.31}$$

Therefore, the unknown coefficients β_n can be calculated as

$$(\forall n = 0,1,...,K-1) \;\; \beta_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\sum_{k=1}^{K} w^k f^k \phi_n(\xi^k)}{\sum_{k=1}^{K} w^k [\phi_n(\xi^k)]} \, . \tag{1.32}$$

This is the same formula as the ones for calculating generalised Fourier coefficients in discrete least-squares approximation. While approximation may require only the first few terms $\beta_n \phi_n$, an interpolation employs all K terms $\beta_n \phi_n$. Hence, the **discrete least-squares approximation, based on linear combination of *K* functions that are orthogonal at *K* distinct sampling points, is equal to interpolation**.

The most important interpolations of this type are

- Fourier interpolation (Chapter 3.4)
- Chebyshev polynomial interpolation (Chapter 2.3)
- Chebyshev rational interpolation (Chapter 3.3)

An advantage of these interpolation methods rely on easier calculation of the coefficients β_n (without solving the linear equation system (1.18)). In addition, Fourier interpolation is suitable to be applied to periodic functions, while Chebyshev interpolations minimises the Runge's phenomenon and the interpolation error (Chapter 2.4).

The disadvantage of interpolation with orthogonal functions may be in the fact that positions of nodes are predetermined by the used set of orthogonal functions.

1.2 SIMPLE INTERPOLATION WITH SHAPE FUNCTIONS

Shape functions $N^k(x)$ in a simple interpolation[1] must satisfy the condition that an interpolation function $\mathcal{F}(x)$ in all nodes ξ^k has the values $f^k = \mathcal{F}(\xi^k)$ (1.1). By setting it into interpolation formula $\mathcal{F} = \Sigma_m f^m N^m$ (1.3) it can be obtained that

$$(\forall\, k = 1,\ldots,K)\; f^k = \sum_{m=1}^{K} f^m N^m(\xi^k) \quad \text{or} \quad \sum_{m=1}^{K} f^m (N^m(\xi^k) - \delta_{mk}) = 0\,. \qquad (1.33)$$

Kronecker's delta symbol δ_{mk} has the value 1 for $m = k$ and the value 0 for $m \neq k$, i.e.,

$$\delta_{mk} = \begin{cases} 1 & \forall m = k \\ 0 & \forall m \neq k \end{cases} \qquad (1.34)$$

Since values f^m may take any set of real numbers, the expression (1.33) results in the fundamental property of shape functions

$$N^m(\xi^k) = \delta_{mk} \quad (k, m = 1,\ldots,K\,). \qquad (1.35)$$

In other words, each shape function $N^m(x)$ must have the value 1 in node ξ^m and the value 0 in all other nodes ξ^k ($k \neq m$).

Shape functions $N^m(x)$ can be determined uniquely for a particular set of coordinate functions and a particular distribution of nodes ξ^k. Once when the shape functions $N^m(x)$ have been determined, it is easy to apply the interpolation formula $\mathcal{F} = \Sigma_m f^m N^m$ (1.3) to perform an interpolation of the function $f(x)$ by using its values $f^k = f(\xi^k)$ in the K prescribed nodes ξ^k.

Shape functions can be found by using

1. Null-points, null-lines or null-planes (Chapter 1.2.1),
2. Coordinate functions, (Chapter 1.2.2),
3. Cramer's rule (Chapter 1.2.3), and
4. Orthogonal functions (Chapter 1.2.4).

[1] Properties of shape functions in higher level interpolation are described in Chapter 2.6.

1.2.1 Null points, null lines and null planes

In most cases shape functions can easily be defined by their *null-points*, *null-lines* or *null-plains*, depending on whether interpolation is provided on a line, on a plane or on a volume. Null-points correspond to nodes, while null-lines and null-planes intersects in nodes.

In Example 1.2, it is shown how polynomial shape function can be defined easily and uniquely by their roots; i.e., by their *null-points*.

Example 1.2. Construction of the shape function. The shape function $N^2(x)$ has the value 1 in node ξ^2 and the value 0 in remaining $K-1$ nodes ξ^k ($k \neq 2$), as shown in Fig. 1.3.

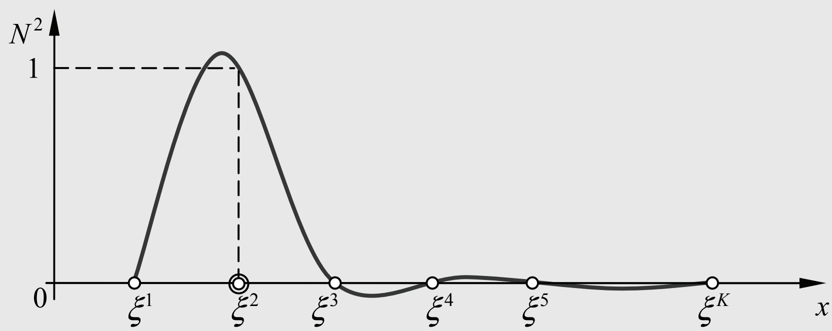

Fig. 1.3. *Example of a shape function*

Shape function $N^2(x)$ may be a polynomial of order $K-1$

$$N^2(x) = \alpha_0^2 + \alpha_1^2 x + \alpha_2^2 (x)^2 + \alpha_3^2 (x)^3 + \alpha_4^2 (x)^4 + ... + \alpha_{K-1}^2 (x)^{K-1} . \quad (1.36)$$

Coefficients α_n^2 are uniquely determined by the conditions

$$N^2(\xi^1) = 0 ,\ N^2(\xi^2) = 1 ,\ N^2(\xi^3) = 0 ,\ \ldots,\ N^2(\xi^K) = 0 . \quad (1.37)$$

Calculation of coefficients α_n^2 can be avoided by expressing the polynomial $N^2(x)$ (1.36) in the form

$$N^2(x) = \frac{(x-\xi^1)(x-\xi^3)(x-\xi^4)(x-\xi^5)...(x-\xi^K)}{(\xi^2-\xi^1)(\xi^2-\xi^3)(\xi^2-\xi^4)(\xi^2-\xi^5)...(\xi^2-\xi^K)} . \quad (1.38)$$

Polynomials expressed in this form are called *Lagrange polynomials* and are described in Chapter 2.2.